# Fermionic Toric Code

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Abstract: This report provides a comprehensive theoretical analysis of the fermionic toric code, a paradigmatic model for understanding topological order in systems of interacting fermions. We begin by establishing the mathematical framework necessary for describing fermionic phases, including super vector spaces and the crucial roles of fermion parity and time-reversal symmetry. After a brief review of the standard bosonic toric code to establish a baseline, we present two formal definitions of the fermionic toric code: the first, a constructive string-net model built upon $\mathbb{Z}_2$-graded fusion rules; the second, an emergent description arising from a physically motivated model of interacting Majorana zero modes on superconducting islands. A detailed analysis of the model's topological properties reveals a ground state degeneracy identical to its bosonic counterpart, but with a fundamentally different anyon content and braiding statistics, which we characterize using the modular $S$ and $T$ matrices and an effective $K$-matrix Chern-Simons theory. We frame the fermionic toric code as the canonical example of a $\mathbb{Z}_2$ topological order enriched by fermion parity symmetry ($\mathbb{Z}_2^f$), leading to a phenomenon of statistical inversion among its quasiparticle excitations. Finally, we discuss the model's strengths and applications in the context of fault-tolerant quantum computation, where it serves as the foundation for Majorana surface codes, representing a promising synergy between physical hardware and logical code design.

## Contents



## Contents

# 1 Introduction: from bosonic to fermionic topological order

## 1.1 The paradigmatic $\mathbb{Z}_2$ topological order

The study of topological phases of matter in $(2 + 1)$ dimensions often begins with Kitaev's toric code, a quintessential model realizing $\mathbb{Z}_2$ topological order in a system of localized spins (qubits) [1, 2]. Defined on a two-dimensional square lattice, its Hamiltonian is a sum of mutually commuting local projectors, known as star and plaquette operators. This exact solvability allows for a complete characterization of its properties: a gapped energy spectrum, a topological ground state degeneracy on manifolds with non-trivial topology (e.g., four-fold on a torus), and deconfined anyonic excitations. These excitations, an electric charge $e$ and a magnetic flux $m$, are local bosons with respect to themselves but exhibit mutual semionic statistics—braiding an $e$ particle around an $m$ particle imparts a phase of $-1$ to the wavefunction. Their composite, $\varepsilon = e \times m$, is a fermion. The toric code thus provides the simplest lattice realization of a $\mathbb{Z}_2$ gauge theory and serves as a foundational blueprint for topological quantum error correction [1, 2].

## 1.2 The richer world of fermionic phases

While the toric code provides a complete picture for a bosonic system, the landscape of topological phases becomes considerably richer when the underlying constituents are fermions [3, 4]. This increased complexity arises from two fundamental aspects of fermionic systems. First, the principle of locality is altered by Fermi-Dirac statistics; the anticommutation relations of fermion operators are inherently non-local. Second, and relatedly, any closed system of fermions possesses an unremovable $\mathbb{Z}_2^f$ symmetry corresponding to the conservation of fermion parity, $P_f = (-1)^{\hat{N}_f}$, where $\hat{N}_f$ is the total fermion number operator [5]. This symmetry acts as a superselection rule, partitioning the Hilbert space into even and odd parity sectors.

Consequently, a fermionic topological phase is not merely a topological order but a topological order intertwined with this fundamental symmetry. This leads to the concept of a **Symmetry-Enriched Topological (SET) phase** [6–8]. An SET phase is characterized not only by its intrinsic anyon content but also by how those anyons transform under the action of a global symmetry. The fermionic toric code can be understood precisely in this context: it is not a fundamentally new topological order, but rather the familiar $\mathbb{Z}_2$ topological order enriched by the ever-present $\mathbb{Z}_2^f$ fermion parity symmetry.

The fermionic toric code represents the most fundamental incorporation of fermionic statistics into the $\mathbb{Z}_2$ gauge theory framework. It retains the same topological ground state degeneracy and topological entanglement entropy as its bosonic precursor, yet the internal structure—specifically, the statistics of its elementary excitations—is profoundly altered [4]. This makes it the ideal pedagogical model for exploring the transition from bosonic to fermionic topological order and for understanding the core principles of symmetry enrichment.

## 2 Mathematical framework for fermionic systems

To properly describe fermionic topological phases, one must adopt a mathematical framework that inherently respects the properties of fermions. This involves moving from standard vector spaces to super vector spaces and carefully defining the action of fundamental symmetries.

### 2.1 Super vector spaces and the fermionic fock space

A fermionic system is naturally described by a $\mathbb{Z}_2$-graded Hilbert space. This structure is formalized by the concept of a super vector space [9]. A super vector space $V$ is a vector space that decomposes into a direct sum of two subspaces: an "even" (bosonic) subspace $V_0$ and an "odd" (fermionic) subspace $V_1$.

$$V = V_0 \oplus V_1$$

The fermionic Fock space $\mathcal{F}$ exhibits this structure directly. The subspace spanned by basis states with an even number of fermions constitutes $\mathcal{F}_0$, while the subspace spanned by states with an odd number of fermions constitutes $\mathcal{F}_1$. A vector $v \in V$ is said to be ***homogeneous*** if it belongs entirely to either $V_0$ or $V_1$. Its parity or degree, denoted $|v|$, is 0 if $v \in V_0$ and 1 if $v \in V_1$.

This grading has profound consequences for multilinear algebra. The tensor product of two super vector spaces $V$ and $W$ is itself a super vector space, with the grading defined as:

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$$
$$(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$$

This implies that the degree is additive: $|v \otimes w| = |v| + |w| \pmod 2$. The crucial departure from standard vector spaces appears in the braiding isomorphism $c_{V,W} : V \otimes W \to W \otimes V$. It is defined for homogeneous vectors $v \in V$ and $w \in W$ as:

$$c_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

This is the Koszul sign rule, which yields $-1$ phase only when both $w$ and $v$ are odd [9].

### 2.2 Fermion parity symmetry

The fermion parity operator is defined as $\hat{P}_f = (-1)^{\hat{N}_f}$, where $\hat{N}_f = \sum_i c_i^\dagger c_i$ is the total fermion number operator. For any closed system of fermions described by a local Hamiltonian, this operator is conserved:

$$[\hat{H}, \hat{P}_f] = 0 \tag{2.1}$$

This conservation is not an accidental symmetry but a fundamental consequence of locality in a fermionic system [5]. A single fermion creation or annihilation operator, $c_i^\dagger$ or $c_i$, is an inherently non-local object. To satisfy the canonical anticommutation relations $\{c_i, c_j^\dagger\} = \delta_{ij}$ and $\{c_i, c_j\} = 0$, the operator must effectively "know" the parity of all other occupied modes in the system, a fact often made explicit in mappings to spin systems via the Jordan-Wigner string. A local Hamiltonian term must be constructed from operators with local support. A term linear in a fermion operator, like $H_i = c_i + c_i^\dagger$, would violate this principle. Therefore, any local term in a fermionic Hamiltonian must be composed of an even number of fermion operators, ensuring it commutes with $\hat{P}_f$. This elevates fermion parity conservation to a fundamental superselection rule, meaning no physical operation can create a superposition of states with different fermion parities.

### 2.3 Time-reversal symmetry in fermionic systems

Time-reversal symmetry, implemented by an operator $\mathcal{T}$, plays a pivotal role in classifying fermionic phases, most notably topological insulators. In quantum mechanics, $\mathcal{T}$ must be an anti-unitary operator by wigner's theorem [10]. This requirement stems from the need to preserve the canonical commutation relations, e.g., $[\hat{x}, \hat{p}] = i\hbar$, under the transformation $\hat{x} \to \hat{x}$ and $\hat{p} \to -\hat{p}$. For the commutator to transform correctly ($[\hat{x}, -\hat{p}] = -i\hbar$), the operator must also transform $i \to -i$, which is the definition of an anti-unitary operator: $\mathcal{T}(\alpha|\psi\rangle) = \alpha^* \mathcal{T}|\psi\rangle$.

For a spin-1/2 fermion described by creation operators $c_{i,\sigma}^\dagger$, the action of time-reversal is given by [11]:

$$\mathcal{T} c_{i,\uparrow} \mathcal{T}^{-1} = c_{i,\downarrow} \quad , \quad \mathcal{T} c_{i,\downarrow} \mathcal{T}^{-1} = -c_{i,\uparrow}$$

A crucial property follows from applying the operator twice:

$$\mathcal{T}^2 c_{i,\uparrow} (\mathcal{T}^2)^{-1} = \mathcal{T}(c_{i,\downarrow})\mathcal{T}^{-1} = -c_{i,\uparrow}$$

This implies that for a state with an odd number of fermions (a half-integer total spin), the time-reversal operator squares to minus the identity:

$$\mathcal{T}^2 = -1 \quad \text{(for half-integer spin systems)}$$

This property leads to **Kramers' Theorem**, a cornerstone of fermionic condensed matter physics [10, 12].

**Theorem 2.1: Kramers' Theorem**

For any eigenstate $|\psi\rangle$ of a time-reversal symmetric Hamiltonian ($[\mathcal{H}, \mathcal{T}] = 0$) in a system where $\mathcal{T}^2 = -1$, the state $\mathcal{T}|\psi\rangle$ is also an eigenstate with the same energy. Furthermore, these two states are orthogonal: $\langle\psi|\mathcal{T}\psi\rangle = 0$.

**(*Proof*)**
The orthogonality follows from the anti-unitary property of $\mathcal{T}$:

$$\langle\psi|\mathcal{T}\psi\rangle = \langle\mathcal{T}^2\psi|\mathcal{T}\psi\rangle^* = \langle-\psi|\mathcal{T}\psi\rangle^* = -\langle\psi|\mathcal{T}\psi\rangle^* \implies \langle\psi|\mathcal{T}\psi\rangle = 0$$

■

This guarantees that every energy level in such a system is at least two-fold degenerate. This "Kramers degeneracy" is protected by time-reversal symmetry and can only be lifted by a perturbation, such as an external magnetic field, that breaks it.

## 3 Review of the bosonic toric code

To appreciate the unique features of the fermionic toric code, we first briefly review its bosonic counterpart.

### 3.1 The stabilizer hamiltonian

The model is defined on a $L \times L$ square lattice with periodic boundary conditions (a torus). A spin-1/2 degree of freedom (a qubit) resides on each edge of the lattice, for a total of $2L^2$ qubits. The Hamiltonian is a sum of two types of mutually commuting operators [2, 13]:

$$H_{\text{TC}} = -J_e \sum_v A_v - J_m \sum_p B_p$$

where the sum is over all vertices $v$ and plaquettes $p$ of the lattice.

- The **star operator** $A_v$ is a product of Pauli-$\sigma^z$ operators on the four edges meeting at vertex $v$:
$$A_v = \prod_{i \in \text{star}(v)} \sigma_i^z$$
- The **plaquette operator** $B_p$ is a product of Pauli-$\sigma^x$ operators on the four edges forming the plaquette $p$:
$$B_p = \prod_{j \in \text{plaq}(p)} \sigma_j^x$$

A key property is that all operators in the Hamiltonian commute: $[A_v, A_{v'}] = [B_p, B_{p'}] = [A_v, B_p] = 0$. This makes the model exactly solvable.

### 3.2 Ground state and excitations

The ground state subspace consists of states $|\psi_G\rangle$ that are simultaneous $+1$ eigenstates of all stabilizer operators:

$$A_v|\psi_G\rangle = +1|\psi_G\rangle \quad \forall v \quad \text{and} \quad B_p|\psi_G\rangle = +1|\psi_G\rangle \quad \forall p$$

On a torus, there are global constraints ($\prod_v A_v = 1$ and $\prod_p B_p = 1$), leaving $2L^2 - 2$ independent stabilizers. This results in a $2^{2L^2-(2L^2-2)} = 4$-dimensional ground state subspace, which can encode two logical qubits [1, 2].

Excitations correspond to violations of the stabilizer conditions.

- An **electric charge** ($e$) resides on vertex $v$ if $A_v = -1$.
- A **magnetic flux** ($m$) resides on plaquette $p$ if $B_p = -1$.

These excitations are created in pairs at the endpoints of string operators. A string of $\sigma^x$ operators along a path on the lattice creates a pair of $e$ charges at its ends. A string of $\sigma^z$ operators along a path on the dual lattice creates a pair of $m$ fluxes at its ends.

### 3.3 Anyonic braiding statistics

The excitations of the toric code are not fundamental bosons or fermions; they are anyons. Their statistical properties are summarized as follows:

- **Self-statistics:** Both $e$ and $m$ are bosons. Braiding two identical particles results in a trivial phase factor of $+1$.
- **Mutual-statistics:** When an $e$ particle is moved in a closed loop around an $m$ particle, the wavefunction acquires a phase factor of $-1$. This is a hallmark of their non-trivial mutual statistics [2].
- **Fusion:** The fusion product of an $e$ and an $m$ particle creates a composite particle, the dyon $\varepsilon = e \times m$. This composite particle is a fermion, as braiding two $\varepsilon$ particles is equivalent to braiding their constituent parts, resulting in a phase of $(-1) \times (-1) \times (-1) = -1$.

The complete set of anyons is $\{1, e, m, \varepsilon\}$, where 1 is the vacuum. Their braiding and fusion properties can be fully described by the modular $S$ and $T$ matrices. For the basis $(1, e, m, \varepsilon)$, these are:

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad , \quad T = \mathrm{diag}(1, 1, 1, -1)$$

The matrix $T$ encodes the self-statistics (topological spin, $\theta_a$), with $T_{aa} = e^{2\pi i \theta_a}$. The matrix $S$ encodes the mutual statistics from braiding particle $a$ around particle $b$.

## 4 Formal definitions of the fermionic toric code

The fermionic toric code phase is a robust concept, underscored by the fact that it can be realized through distinct theoretical constructions. One approach is a “top-down” method based on abstract algebraic structures (string-net models), which builds the desired properties into the Hamiltonian by design. Another is a “bottom-up” approach starting from a physically plausible model of interacting Majorana fermions, where the topological order emerges in a low-energy effective theory.

### 4.1 Construction I: the string-net model with fermions

This construction, developed by Gu, Wang, and Wen, provides an exactly solvable lattice model that realizes the fermionic toric code by generalizing the string-net formalism to fermionic systems [3].

### 4.1.1 Lattice and degrees of freedom

The model is defined on a trivalent lattice, typically a honeycomb lattice, to simplify the fusion rules at each vertex. The degrees of freedom are twofold:

1. **Fermionic modes:** At each vertex $v$ of the lattice, there is a fermionic mode with creation and annihilation operators $c_v^\dagger$ and $c_v$, satisfying $\{c_v, c_{v'}^\dagger\} = \delta_{vv'}$ [3].

2. **String types:** On each link $e$, there is a two-level system (qubit) whose states, $|0\rangle_e$ and $|1\rangle_e$, represent the absence or presence of a string of a particular type. These can be represented by Pauli operators $\sigma_e^z, \sigma_e^x$.

A crucial additional piece of structure is a **branching structure**: a fixed orientation (arrow) on each link such that at any vertex, the arrows are not all incoming or all outgoing. This structure is necessary to define signs in the Hamiltonian terms unambiguously [3, 4]. A possible branching structure is shown in Figure 1.

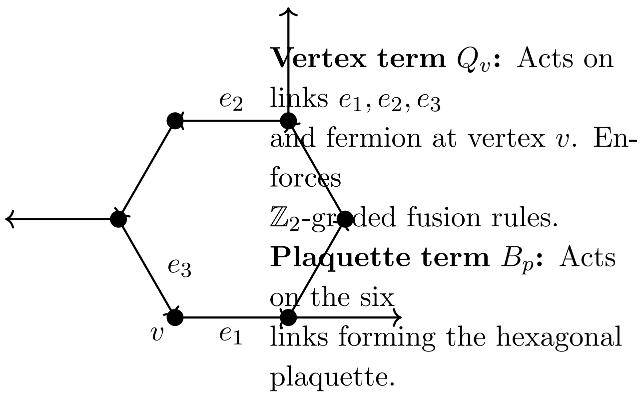


**Figure 1**. Schematic of the fermionic toric code on a honeycomb lattice. Fermionic modes reside on the vertices (black dots), while string-type qubits reside on the links. A branching structure (arrows) is fixed. The vertex operator $Q_v$ acts on the three links and one vertex incident to it, while the plaquette operator $B_p$ acts on the six links of a hexagon.

### 4.1.2 The Hamiltonian from $\mathbb{Z}_2$-Graded Fusion Rules

The Hamiltonian is a sum of commuting projectors

$$H_{fTC} = -\sum_v Q_v - \sum_p B_p. \tag{4.1}$$

The operators are constructed based on a $\mathbb{Z}_2$-graded version of the $\mathbb{Z}_2$ fusion category [3, 4]. The fusion of two strings of type '1' can result in a vacuum string '0' in either a bosonic (fermion-parity even) or fermionic (fermion-parity odd) channel. The latter implies the creation of a physical fermion at the fusion vertex. This is encoded in the fusion coefficients $N_{ijk}^{b/f}$, where for this model, the only non-trivial fermionic channel is $N_{110}^f = 1$ [3].

- **Vertex Operator $Q_v$:** This operator enforces the fusion rules at each vertex $v$. It is a projector that depends on the state of the three incident links $(\sigma_1^z, \sigma_2^z, \sigma_3^z)$ and the fermion occupation number at the vertex ($\hat{n}_v = c_v^\dagger c_v$). Its form is determined by the fermion number function $N_f(\sigma_1^z, \sigma_2^z, \sigma_3^z)$, which is 1 for the configuration corresponding to $1 \otimes 1 \to 0_f$ and 0 otherwise. For a vertex with two incoming arrows and one outgoing arrow, $Q_v$ projects onto the state where $\hat{n}_v = N_f(\sigma_{\text{in}1}^z, \sigma_{\text{in}2}^z, \sigma_{\text{out}}^z)$. This explicitly ties the string configuration to the local fermion parity [4].

- **Plaquette Operator $B_p$:** This operator acts on the six links of a hexagonal plaquette. It is a product of operators $\hat{O}(\{\sigma_b \in p_z\})$, which are more complex than the simple product of $\sigma^x$ in the bosonic case. These operators act on the string degrees of freedom and can be interpreted as creating, moving, and annihilating strings around the plaquette, with phases determined by the fermionic associativity relations (F-symbols) of the underlying graded category [3].

The ground state of this Hamiltonian is an equal-weight superposition of all closed string-net configurations, where each configuration is "decorated" with fermions on the vertices as dictated by the graded fusion rules [3].

#### 4.1.3 Relation to spin TQFT

The low energy effective theory for the bosonic toric code is the Chern-Simons theory

$$S_{\text{CS}} = \frac{K_{IJ}}{4\pi} \int a_I \wedge da_J. \tag{4.2}$$

with $K$-matrix

$$K^{\text{TC}} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{4.3}$$

This can be seen as follows: The $K$-matrix is not an abstract postulate but a direct encoding of the physical braiding statistics of the anyonic excitations. We can derive its form by introducing two emergent $U(1)$ gauge fields, $a_e$ and $a_m$, whose fluxes correspond to the $e$ (electric) and $m$ (magnetic) anyons, respectively. In the basis of these fields, the charge vectors for the elementary anyons are $\mathbf{l}_e = (1,0)^T$ and $\mathbf{l}_m = (0,1)^T$. The braiding statistics are related to the inverse of the $K$-matrix by two fundamental formulas:

- **Self-statistics (Topological Spin):** The statistical angle $\theta_a$ for an anyon with charge vector $\mathbf{l}_a$ is given by $\theta_a = \pi \mathbf{l}_a^T K^{-1} \mathbf{l}_a \pmod{2\pi}$. The corresponding topological spin is $e^{i\theta_a}$.

- **Mutual statistics:** The statistical angle $\theta_{ab}$ acquired when braiding anyon $a$ around anyon $b$ is given by $\theta_{ab} = 2\pi \mathbf{l}_a^T K^{-1} \mathbf{l}_b \pmod{2\pi}$.

For the bosonic toric code, we apply these rules to the known statistics:

1. The $e$ particle is a boson, so its self-statistics phase is $+1$, meaning $\theta_e = 0 \pmod{2\pi}$. This implies $\pi \mathbf{l}_e^T K^{-1} \mathbf{l}_e = \pi (K^{-1})_{11} = 0 \pmod{2\pi}$, which requires $(K^{-1})_{11}$ to be an even integer. The simplest choice is $(K^{-1})_{11} = 0$.

2. Similarly, the $m$ particle is a boson, so $\theta_m = 0 \pmod{2\pi}$. This implies $\pi \mathbf{l}_m^T K^{-1} \mathbf{l}_m = \pi (K^{-1})_{22} = 0 \pmod{2\pi}$, requiring $(K^{-1})_{22}$ to be an even integer. We choose $(K^{-1})_{22} = 0$.

3. Braiding an $e$ particle around an $m$ particle yields a phase of $-1$, corresponding to a mutual statistical angle of $\theta_{em} = \pi$. This implies $2\pi \mathbf{l}_e^T K^{-1} \mathbf{l}_m = 2\pi (K^{-1})_{12} = \pi \pmod{2\pi}$, which fixes $(K^{-1})_{12} = 1/2$. By symmetry of $K^{-1}$, we also have $(K^{-1})_{21} = 1/2$.

Assembling these results gives the inverse $K$-matrix:

$$(K^{\mathrm{TC}})^{-1} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}. \tag{4.4}$$

Inverting this matrix directly yields the established $K$-matrix for the bosonic toric code, $K^{\mathrm{TC}} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

Likewise, the fermionic toric code hosts a spin TQFT as its low energy EFT, where

$$K^{f\mathrm{TC}} = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}, \tag{4.5}$$

in the Chern-Simons action (4.2). The emergence of this specific Chern-Simons theory is a direct consequence of the altered anyon statistics in the fermionic model. In the fermionic toric code, the electric charge excitation is identified with the fundamental fermion of the system. Let us label this excitation $f$. The magnetic flux $m$ remains a boson. Their mutual statistics are still semionic. Applying the same derivation as above:

1. The $m$ particle is a boson: $\theta_m = 0 \pmod{2\pi} \implies (K^{-1})_{mm}$ is an even integer (we choose 0).

2. The $f$ particle is a fermion: its self-statistics phase is $-1$, so $\theta_f = \pi \pmod{2\pi}$. This implies $\pi \mathbf{l}_f^T K^{-1} \mathbf{l}_f = \pi (K^{-1})_{ff} = \pi \pmod{2\pi}$, which requires $(K^{-1})_{ff}$ to be an odd integer. Let's choose $(K^{-1})_{ff} = 1$.

3. The mutual statistics are unchanged: $\theta_{mf} = \pi \pmod{2\pi} \implies (K^{-1})_{mf} = 1/2$.

In the basis of physical excitations $(m, f)$, the inverse $K$-matrix is $(K')^{-1} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. Inverting this gives $K' = \begin{pmatrix} -4 & 2 \\ 2 & 0 \end{pmatrix}$. The specific form of the $K$-matrix depends on the choice of basis for the gauge fields. Different bases are related by $SL(2, \mathbb{Z})$ transformations, $W$, which act on the $K$-matrix as $K \to WKW^T$. The matrix $K^{f\mathrm{TC}} = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$ is related to $K'$ by such a transformation and thus describes the same topological phase. The crucial, basis-independent feature is the presence of an odd integer on the diagonal. A bosonic theory, with all diagonal entries of $K$ being even, can never be transformed into a theory with an odd diagonal entry via an $SL(2,\mathbb{Z})$ transformation, proving that this property distinguishes two fundamentally different classes of topological order.

The modification to the diagonal entry of the $K$-matrix is precisely what necessitates a **spin TQFT**. A TQFT is classified as a "spin" TQFT if its partition function on a given spacetime manifold $M$ depends on the choice of a spin structure on $M$. This dependence arises from a fundamental consistency requirement in the path integral formulation of Chern-Simons theory. The path integral involves the factor $e^{iS_{\mathrm{CS}}}$. For the quantum theory to be well-defined, this factor must be unambiguous under certain topological transformations. Specifically, the action evaluated on any closed, oriented 4-manifold $W_4$ must be an integer multiple of $2\pi$. This is the famous Dirac quantization condition extended to the TQFT action. The condition is:

$$\frac{1}{4\pi}\int_{W_4} K_{IJ}F^I \wedge F^J \in 2\pi\mathbb{Z}, \tag{4.6}$$

where $F^I = da^I$ are the field strengths. In topological terms, the integral $\frac{1}{2\pi}\int F^I$ over a 2-cycle represents an integer cohomology class, the first Chern class $c_1(L_I)$ of the corresponding line bundle $L_I$. The condition thus becomes a statement about the intersection form of 2-cycles on the 4-manifold:

$$\frac{1}{2}\sum_{I,J} K_{IJ}\int_{W_4} c_1(L_I) \cup c_1(L_J) \in \mathbb{Z}. \tag{4.7}$$

The term $Q_{IJ} = \int_{W_4} c_1(L_I) \cup c_1(L_J)$ is an integer. The problem arises from the diagonal terms $K_{II}Q_{II}$. For a general oriented 4-manifold, there is no restriction on the parity of $Q_{II}$; it can be an odd integer. If a diagonal element $K_{II}$ is odd, and we choose a manifold and line bundle such that $Q_{II}$ is also odd, the term $\frac{1}{2}K_{II}Q_{II}$ will be a half-integer, violating the quantization condition. The path integral is ill-defined.

This obstruction is removed by restricting the theory to manifolds that admit a spin structure. A deep theorem in topology states that for any spin 4-manifold, the diagonal entries of the intersection form are always even integers, i.e., $\int_W c \cup c \in 2\mathbb{Z}$ for any $c \in H^2(W,\mathbb{Z})$. With this restriction, the term $\frac{1}{2}K_{II}Q_{II}$ is guaranteed to be an integer even if $K_{II}$ is odd, because $Q_{II}$ is now guaranteed to be even. The theory becomes well-defined, but only on the subclass of spin manifolds. This dependence on spin structure is the defining property of a spin TQFT [14]. This entire structure is a beautiful manifestation of the spin-statistics theorem at the level of the effective field theory: the microscopic fermionic nature of the system's constituents forces one of the emergent anyons to be a fermion, which in turn dictates an odd diagonal entry in the $K$-matrix, ultimately requiring the macroscopic, low-energy vacuum description to be sensitive to the global spacetime property that distinguishes fermions from bosons—the spin structure.

### 4.2 Construction II: emergent order from majorana fermions

An alternative and physically motivated route to the fermionic toric code was proposed by Terhal, Hassler, and DiVincenzo [15]. Here, the topological order is not built in by design but emerges as a low-energy effective theory of a system of interacting Majorana fermions.

### 4.2.1 Physical model and hamiltonian

The system consists of a 2D square lattice of mesoscopic superconducting islands. Each island $i$ is assumed to host four Majorana zero modes, denoted $\gamma_{i,a}, \gamma_{i,b}, \gamma_{i,c}, \gamma_{i,d}$ [15]. These four Majorana operators can be combined into two complex fermion modes, e.g., $f_{i,1} = (\gamma_{i,a} + i\gamma_{i,b})/2$ and $f_{i,2} = (\gamma_{i,c} + i\gamma_{i,d})/2$. The setup is depicted in Figure 2.

The Hamiltonian is separated into a large intra-island term $H_0$ and a smaller inter-island perturbation $V$:

$$H = H_0 + V = \sum_i H_{0,i} + \lambda \sum_{\langle i,j \rangle} V_{i,j}$$

- The intra-island term $H_{0,i}$ creates a large energy gap $\Delta_c$ and favors a specific fermion parity configuration on each island. A common choice is [16]:

  $$H_{0,i} = -\Delta_c(2n_{i,ab} - 1)(2n_{i,cd} - 1) = -\Delta_c(i\gamma_{i,a}\gamma_{i,b})(i\gamma_{i,c}\gamma_{i,d})$$

  This term penalizes states where the two fermionic modes on the island have different occupation numbers.

- The perturbation $V$ consists of weak tunneling terms between Majorana fermions on adjacent islands. For example, a term coupling island $i$ and its neighbor $j$ along the $\hat{x}$ direction might be $V_{i,j=\mu+\hat{x}} = i\gamma_{i,c}\gamma_{j,a}$ [16]. These terms are quadratic in Majorana operators.

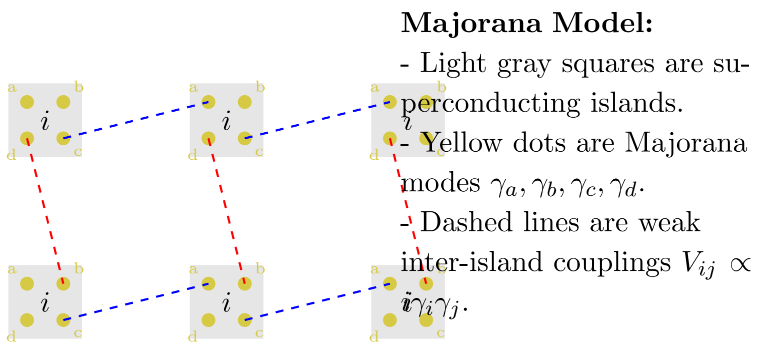


**Figure 2**. The physical layout of the Majorana fermion model. Each island hosts four Majorana modes. Weak quadratic interactions (dashed lines) couple Majoranas on adjacent islands. The toric code emerges as the effective low-energy theory of this system.

### 4.2.2 Emergence of the toric code

In the limit where the coupling $\lambda$ is much smaller than the gap $\Delta_c$, the low-energy physics is confined to the degenerate ground-state manifold of $H_0$. Standard perturbation theory can be used to derive an effective Hamiltonian $H_{\text{eff}}$ acting within this subspace. It is found that the leading non-trivial terms appear at fourth order in perturbation theory, and they take the form of the toric code plaquette operators [15–17].

$$H_{\text{eff}} \approx -\frac{C\lambda^4}{\Delta_c^3} \sum_p A_p^{\text{eff}}$$

Here, $A_p^{\text{eff}}$ is an operator acting on the logical qubits defined by the ground states of the islands surrounding plaquette $p$. This operator is equivalent to the standard toric code plaquette operator $B_p$ (or star operator $A_v$, depending on the mapping). The vertex operators of the toric code are implicitly satisfied by the ground state constraints of $H_0$. This remarkable result shows how the robust topological order of the toric code can emerge from a non-topological, local Hamiltonian of interacting fermions.

Furthermore, this model can be mapped via a 2D Jordan-Wigner transformation to a family of 2D Ising models in a transverse field. The signs of the Ising couplings (ferromagnetic or antiferromagnetic) are determined by the eigenvalues of a set of conserved quantities, which act as a background static $\mathbb{Z}_2$ gauge field. This mapping allows for the study of the model's phase diagram beyond the perturbative limit, revealing a phase transition to a non-topological, polarized phase as $\lambda/\Delta_c$ increases [15].

## 5 Topological properties and excitations

While the fermionic toric code shares the same ground state degeneracy on a torus as its bosonic counterpart (four-fold), its internal topological structure, particularly the nature of its anyonic excitations, is fundamentally different.

### 5.1 Anyon content and statistics inversion

The model supports four distinct superselection sectors, or anyon types, which we can label $\{1, f, m, \psi\}$.

- 1: The vacuum sector, with no excitations.
- $f$: The fundamental fermion. This corresponds to the "electric charge" ($e$) excitation of the bosonic toric code, but it is now identified with the physical fermion of the underlying system.
- $m$: The magnetic flux, or vison. This corresponds to the $m$ particle of the bosonic code and remains a boson.
- $\psi$: The dyon, which is a composite of the fermion and the flux, $\psi = f \times m$.

The most striking difference is the change in statistics of the charge and dyon particles. In the bosonic code, $e$ is a boson and the dyon $\varepsilon$ is a fermion. In the fermionic code, the charge $f$ is a fermion and the dyon $\psi$ is a boson. This phenomenon can be intuitively understood as a form of "fermion condensation." Imagine starting with the bosonic toric code anyons $\{1, e, m, \varepsilon\}$ and then placing this theory in a background filled with a physical fermion, which we also call $f$. The true elementary excitations of the full system are composites of the gauge excitations and the background particles. The physical charge is now the composite of the gauge charge $e$ and the physical fermion $f$. Since a boson ($e$) bound to a fermion ($f$) results in a fermion, the new charge particle is a fermion. The flux $m$ is neutral with respect to fermion parity and remains a boson. The new dyon is a composite of the new charge and the flux: $\psi = (e \times f) \times m$. By associativity, this is equivalent to

$(e \times m) \times f = \varepsilon \times f$. Since both $\varepsilon$ and $f$ are fermions, their composite is a boson. This leads to a complete inversion of statistics for the charge-like and dyon-like excitations.

### 5.2 Fermionic braiding statistics: $S$ and $T$ matrices

This statistical inversion is rigorously captured by the modular data of the theory. The low-energy effective theory of a fermionic topological order is a spin topological quantum field theory (TQFT). For the fermionic toric code, the corresponding Chern-Simons theory is described by a $K$-matrix of the form [3, 4]:

$$K_{fTC} = \begin{pmatrix} \pm 1 & 2 \\ 2 & 0 \end{pmatrix}$$

This is in stark contrast to the K-matrix for the bosonic toric code, $K_{bTC} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$. The presence of an odd integer on the diagonal is a defining feature of a fermionic topological order that cannot be realized in any local bosonic system [3, 4].

The topological spin $\theta_a$ (self-statistics) and the modular $S$-matrix can be computed from the $K$-matrix. For $K_{fTC} = \begin{pmatrix} -1 & 2 \\ 2 & 0 \end{pmatrix}$, the four anyons correspond to integer vectors $\mathbf{l}_a$ in the charge lattice, and their statistics are given by $\theta_a = \frac{1}{2}\mathbf{l}_a^T K^{-1} \mathbf{l}_a \pmod 1$. The resulting topological spins and $S$-matrix are distinct from the bosonic case [3]. For one choice of parameters ($\alpha = -i$ in the string-net model), the topological spins are given by the diagonal of the $T$-matrix as $(e^{2\pi i \theta_a}) = (1, 1, e^{i(3\pi/2)}, e^{i(7\pi/2)})$, and the $S$-matrix is [3]:

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \\ 1 & -1 & i & -i \end{pmatrix}$$

Comparing this to the bosonic case reveals the modified braiding relations, including the appearance of complex phases in the $S$-matrix, which signifies a richer statistical structure. A summary of these differences is presented in Table 1.

## 6 Strengths and applications in fault-tolerant quantum computation

The abstract models of the fermionic toric code provide a concrete blueprint for building quantum error-correcting codes, particularly suited for hardware platforms based on Majorana zero modes.

### 6.1 Majorana stabilizer and surface codes

The fermionic toric code is an example of a **Majorana stabilizer code**. In this formalism, analogous to Pauli stabilizer codes, the codespace is the simultaneous eigenspace of a set of commuting operators. However, the stabilizers are not Pauli strings but products of an even number of Majorana operators [18].

**Table 1**. Comparison of bosonic and fermionic toric codes

| **Feature** | **Bosonic Toric Code** | **Fermionic Toric Code** |
|---|---|---|
| **Degrees of Freedom** | Qubits (spins) on edges | Fermions on vertices / Majoranas on islands |
| **Enriching Symmetry** | None (pure $\mathbb{Z}_2$ gauge theory) | Fermion Parity $\mathbb{Z}_2^f$ |
| **Excitations** | $e$ (electric charge, boson), $m$ (magnetic flux, boson) | $f$ (physical fermion, fermion), $m$ (magnetic flux, boson) |
| **Composite Excitation** | $\varepsilon = e \times m$ (dyon, fermion) | $\psi = f \times m$ (dyon, boson) |
| **Self-Statistics ($T$-matrix diagonal)** | $(1, 1, 1, -1)$ | $(1, 1, e^{i(3\pi/2)}, e^{i(7\pi/2)})$ or similar |
| **Low-Energy TQFT ($K$-matrix)** | $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ | $K = \begin{pmatrix} \pm 1 & 2 \\ 2 & 0 \end{pmatrix}$ or equivalent |

The most direct application for fault-tolerant quantum computation is the **Majorana surface code** [19, 20]. This code is the Majorana analogue of the standard qubit surface code. It can be constructed by tiling a 2D lattice with smaller, elementary Majorana codes, such as the "tetron" (a 4-Majorana code encoding one qubit) or "hexon" (a 6-Majorana code) [18, 21]. This architecture provides a scalable design for protecting quantum information in physical systems based on topological superconductors or other platforms capable of hosting and manipulating Majorana zero modes [21].

The use of Majorana-based codes represents a powerful instance of hardware-software co-design. Standard error correction, such as the surface code on transmon qubits, imposes topological protection on physical systems that are not inherently topological. In contrast, Majorana-based platforms have intrinsic topological properties at the hardware level, namely the non-Abelian braiding of Majorana zero modes. The Majorana surface code leverages this by combining two layers of topological protection: the logical encoding of the code protects against local errors (like quasiparticle poisoning or measurement errors), while the physical nature of the Majoranas provides a native set of topologically protected operations that can be used to implement logical gates more efficiently. This synergy between the physical hardware and the logical code structure is a key advantage of this approach. For instance, many logical Clifford gates can be implemented with "zero time overhead" using techniques like lattice surgery, which are adapted for the fermionic nature of the code [21, 22]. The fundamental operations, such as measuring the joint parity of four Majorana operators, are more powerful than the single-qubit gates available in typical qubit platforms, leading to a more integrated and potentially more efficient fault-tolerant architecture.

### 6.2 Advantages and disadvantages

Compared to the standard bosonic toric code implemented on qubits, fermionic and Majorana codes offer a distinct set of advantages and challenges.

#### 6.2.1 Advantages

- **Lower Resource Overhead:** Certain constructions of Majorana codes, particularly Majorana color codes (which are concatenations of Majorana surface codes with smaller Majorana codes), can achieve lower stabilizer weights and require fewer physical Majorana modes per logical qubit for a given code distance compared to their bosonic counterparts. This translates to easier syndrome measurements and a higher encoding rate [21].

- **Efficient Gate Implementation:** The structure of Majorana codes is well-suited for efficient gate implementation. The entire Clifford group can be implemented via braiding of twist defects and lattice surgery techniques, often with zero time overhead, which is a significant advantage over many bosonic code schemes that require sequences of operations or magic state distillation for some Clifford gates [21, 22].

- **Direct Fermion Simulation:** These codes are naturally suited for simulating fermionic systems, as they are built from fermionic degrees of freedom. This avoids the overhead associated with fermion-to-qubit mappings like the Jordan-Wigner or Bravyi-Kitaev transformations that are necessary when simulating fermions on a qubit-based quantum computer [23, 24].

#### 6.2.2 Disadvantages

- **Hardware Realization:** The primary bottleneck is the experimental challenge of creating, controlling, and braiding stable, low-error Majorana zero modes in physical systems. While significant progress has been made in topological superconductors, building large-scale, fault-tolerant devices remains a formidable long-term goal [21].

- **Measurement Complexity:** Although stabilizer weights may be lower, the physical implementation of a fault-tolerant, high-fidelity measurement of the joint parity of four or more Majorana operators is a non-trivial experimental task that is crucial for error correction cycles.

- **Ancillary Systems:** While native for fermionic simulation, interfacing these codes with algorithms that require both qubits and fermions, or connecting them to more conventional qubit-based quantum processors, introduces additional complexity [23].

## 7 Conclusion

The fermionic toric code stands as a cornerstone model in the theory of topological phases of matter. It serves as the canonical example of a $\mathbb{Z}_2$ topological order enriched by the fundamental fermion parity symmetry ($\mathbb{Z}_2^f$). This enrichment, while preserving the topological ground state degeneracy of its bosonic counterpart, fundamentally redefines the nature of the elementary excitations. The charge-like particle becomes a fermion and the dyon becomes a boson, a statistical inversion that is cleanly captured by the modular data of the underlying topological field theory.

The robustness of this phase is highlighted by its dual theoretical origins: it can be constructed exactly as a string-net model with graded fusion rules, and it also emerges as the low-energy effective theory of a physically motivated system of interacting Majorana fermions. This convergence underscores its relevance not just as a theoretical tool but as a potential physical reality.

In the realm of quantum information, the fermionic toric code provides the theoretical foundation for Majorana surface codes, a leading architecture for fault-tolerant quantum computation. This approach offers a compelling paradigm of hardware-software co-design, where the intrinsic topological properties of Majorana zero modes are synergistically combined with the logical protection of a topological code. While significant experimental challenges remain in realizing such hardware, the potential advantages in resource efficiency and native gate operations make the fermionic toric code and its descendants a vital and promising avenue in the quest for a scalable quantum computer.

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